Renaissance Art

Intersecting Parallel Lines:

Projective Geometry and its Applications

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MIT PRIMES Circle

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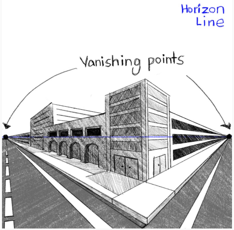
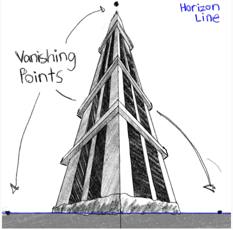
Projective Geometry in Art

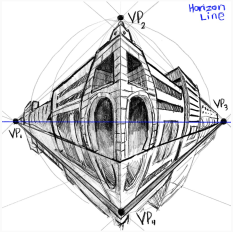
one-point



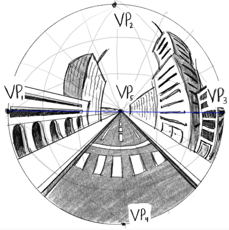
two-point



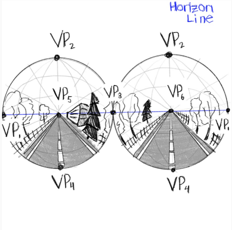
three-point



four-point



five-point



six-point

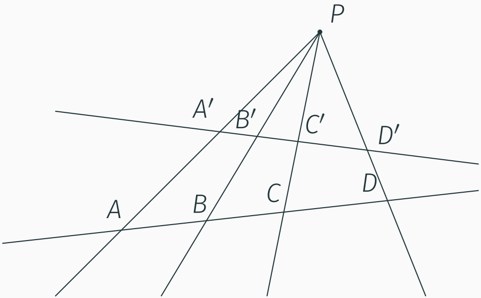
Projective Geometry fundamentals

Perspectivity and Ideal Point

Perspectivity with respect to a point

is the mapping of points A, B, C, D on one line and the mapping of points A', B', C', D' on another line.

Points A, B, C, D and A', B', C', D' are related by a perspectivity with respect to point P.



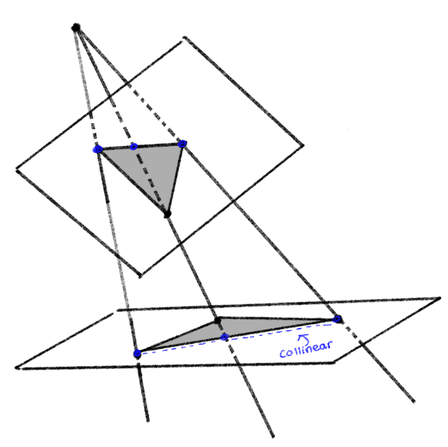
Definition

An *Ideal point* is a point at infinity where parallel line meet.

Collineation

Definition

A collineation is a one-to-one mapping from one projective space to another, or from a projective space to itself, such that the images of collinear points remain collinear after transformations.

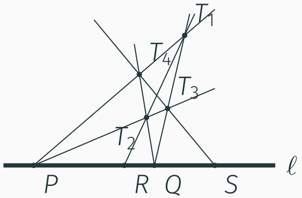


Harmonic Sets and Cross Ratios

Harmonic Set

Four distinct points form a harmonic set, denoted $(P, Q; R, S)$ if and only if they are collinear and their 6 lines form a complete quadrangle.

Cross-ratios: the ratio of four distinct collinear points. $\dfrac{\frac{PR}{QR}}{\frac{PS}{QS}}$



*A Quadrangle Construction

Cross ratio of $(P, Q; R, S)$

Homogeneous Coordinates

- $A(x, y, z) = \lambda(x, y, z)$
- $(x, y, 1) \longrightarrow$ regular point corresponding with Euclidean geometry
- $(x, y, 0) \longrightarrow$ a point at infinity

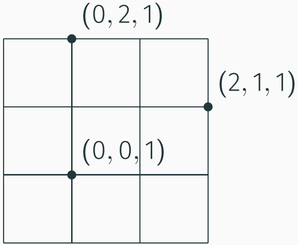


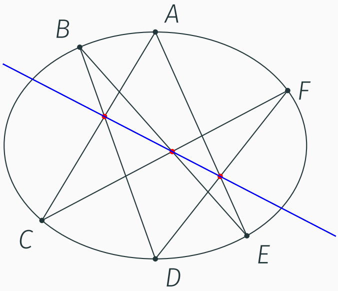
Figure 1: The projective plane

Definition

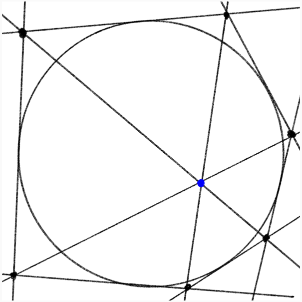
$\mathbb{P}^n = \{(x_0, ..., x_n) : x_0, ..., x_n$ are not all 0 and $(x_0, ...x_n) = \lambda(x_0, ...x_n)\}$.

Duality

A formalization of the symmetry of the roles played by points and lines in the definitions and theorems of projective planes



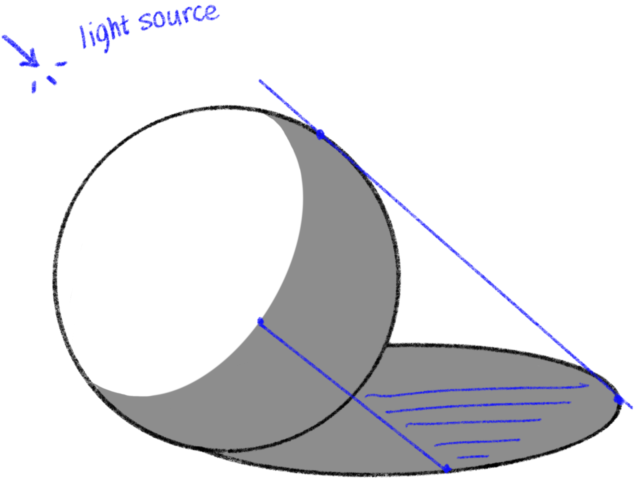
Pascal's Theorem



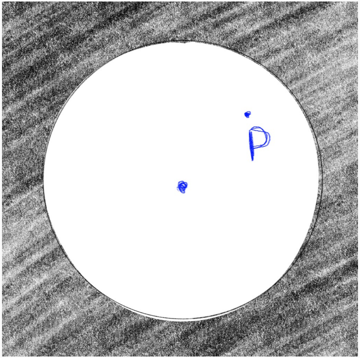
Brianchon's Theorem

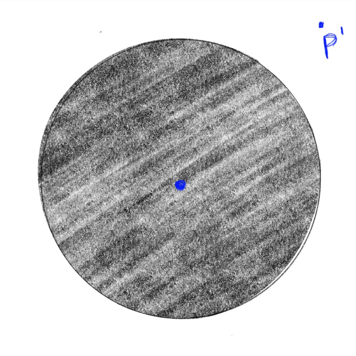
Projective Transformations

Shadows are a form of projective transformations

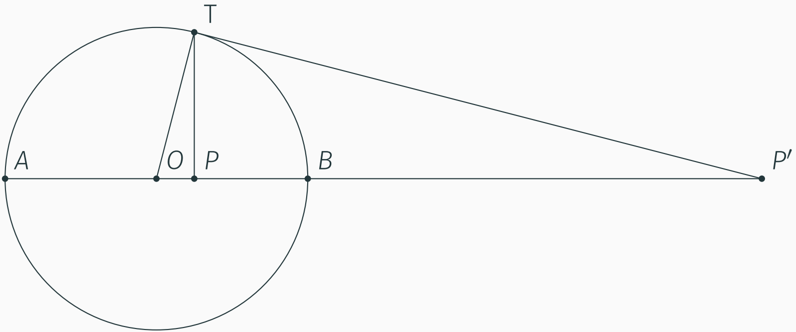


Circle Inversion



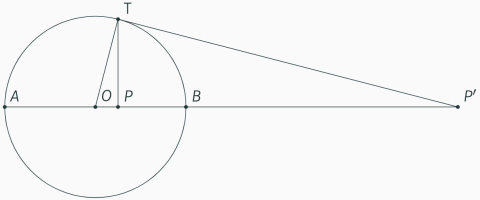


$\overline{OP} \cdot \overline{OP'} = r^2$



Theorem

If P is a point in the diameter AB of a circle with center O, and P' is the inverse of P with respect to this circle, then the cross ratio of $(A, B; P, P') = -1$, i.e., four distinct points A, B, P, P' form a *harmonic set.*



Proof

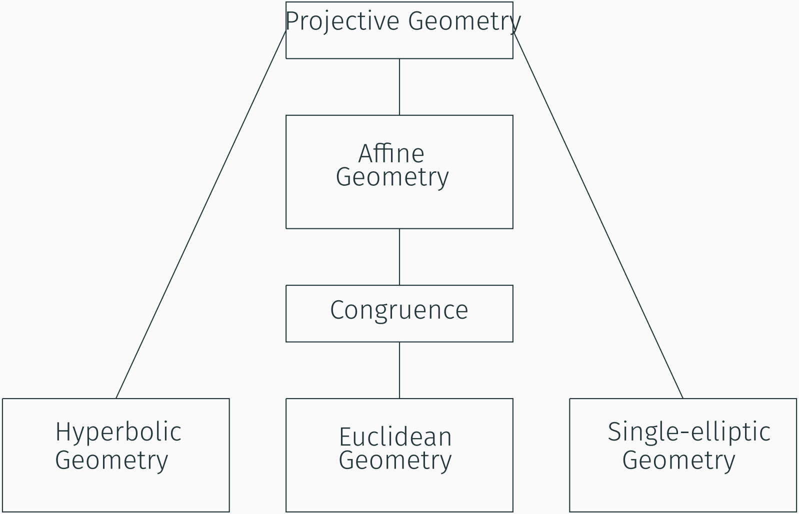
$$\frac{\frac{AP}{PB}}{\frac{AP'}{P'B}} = \frac{\frac{r+OP}{r-OP}}{\frac{OP'+r}{OP'-r}} = \frac{(OP\cdot OP') - (OP\cdot r) + (OP'\cdot r) - r^2}{-(OP\cdot OP') + (OP\cdot r) - (OP'\cdot r) + r^2}$$

$$(OP\cdot OP') = r^2$$

$$\frac{-(OP\cdot r) + (OP'\cdot r)}{(OP\cdot r) - (OP'\cdot r)} = -1$$

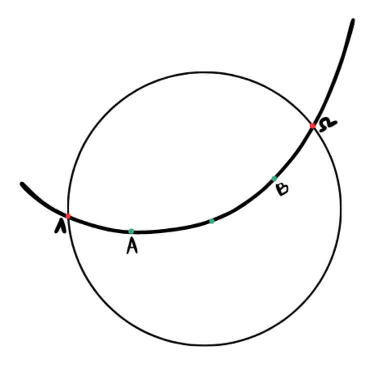
Sub-Geometries

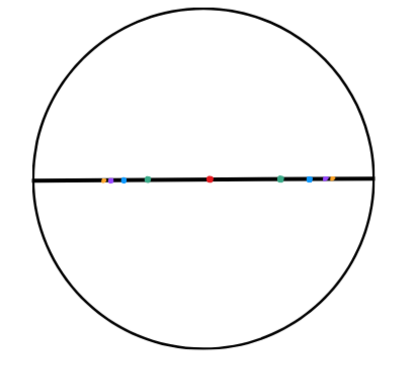
Relation of Geometries Map



Hyperbolic Geometry

study of surfaces with a curvature of -1

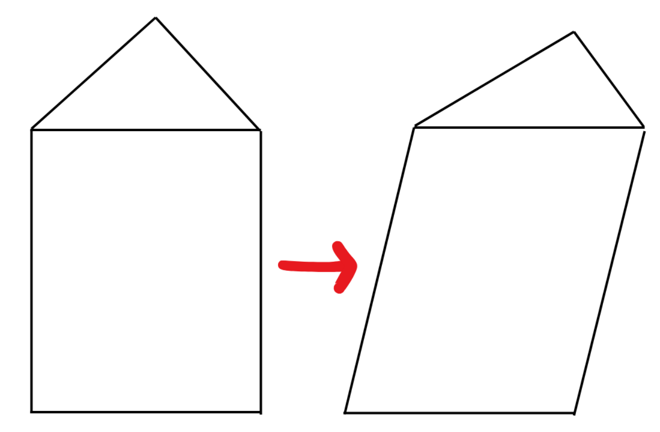




Affine Geometry

As affine geometry does not account for angle or distance metrics, we can say that affine geometry is a sub - geometry of projective geometry.

Euclidean geometry is a sub-geometry of Affine geometry.



Applications

CAD: Computer Aided Design

Animation

Conclusions

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Questions?

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